Last time:

$$\frac{d\underline{x}}{dt} = f(\underline{x}, \underline{u})$$

Solve using the computer.

Recall definition of the derivative: $\frac{dy}{dt} \equiv \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{y(t) - y(t-\Delta t)}{\Delta t}$$

What happens if Δt doesn't go to 0? Taylor series expand.

$$y(t-\Delta t) = y(t) - \Delta t \frac{dy}{dt} + \Delta t^2 \frac{d^2y}{dt^2} \dots$$

$$y(t) - y(t-\Delta t) = \Delta t \frac{dy}{dt} - \Delta t^2 \frac{d^2y}{dt^2} + \dots$$

$$\frac{y(t) - y(t-\Delta t)}{\Delta t} = \frac{dy}{dt} - \Delta t \frac{d^2y}{dt^2} + \dots \Rightarrow \boxed{\frac{y(t) - y(t-\Delta t)}{\Delta t} = \frac{dy}{dt} + O(\Delta t)}$$

* $O(\Delta t)$ is truncation error.
* 1^{st} Order approximation of derivative. As Δt gets smaller, approximation gets better. But, if Δt gets too small, run into precision problems on the computer (rounding error).

* Can increase the Order of approximation by using more data points.

$$\frac{\frac{3}{2}y(t) - 2y(t-\Delta t) + \frac{1}{2}y(t-2\Delta t)}{\Delta t} = \frac{dy}{dt} + O(\Delta t^2)$$

$$\frac{\frac{11}{6}y(t) - 3y(t-\Delta t) + \frac{3}{2}y(t-2\Delta t) - \frac{1}{3}y(t-3\Delta t)}{\Delta t} = \frac{dy}{dt} + O(\Delta t^3)$$

$\underline{\text{Ex}}$ $y(t) = \cos(t)$ $\quad \dot{y} = \frac{dy}{dt} = -\sin(t)$ $\quad \dot{y}(t=0) = 0$

$\Delta t = 0.001$

$\underline{1^{st} \text{ Order:}}$ $\dot{y} = 0.0005$

$\underline{2^{nd} \text{ Order:}}$ $\dot{y} = 2.5 \times 10^{-10}$

$\underline{3^{rd} \text{ Order:}}$ $\dot{y} = -2.5 \times 10^{-10}$

Approximated the derivative. What about $f(t, y, u)$?

Again, Taylor series expand.

$$f(t-\Delta t, y(t-\Delta t), u(t-\Delta t)) = f(t, y(t), u(t)) - \Delta t \frac{\partial f}{\partial t} - \Delta y \frac{\partial f}{\partial y} - \Delta u \frac{\partial f}{\partial u} + \ldots$$

$$f(t-\Delta t, y(t-\Delta t), u(t-\Delta t)) = f(t, y(t), u(t)) - \Delta t \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial u}\frac{du}{dt} \right) + \ldots$$
$$= f(t, y(t) + u(t)) + O(\Delta t)$$

$$\boxed{\text{So, to } 1^{st} \text{ order: } f(t, y(t), u(t)) = f(t-\Delta t, y(t-\Delta t), u(t-\Delta t))}$$

Differential equation: $\frac{dy}{dt} = f(t, y(t), u(t))$

$\underline{\text{nomenclature:}}$ t^n = current time

$t^{n-1} = t^n - \Delta t$

$y^n = y(t^n)$ $\quad y^{n-1} = y(t^{n-1})$

$u^n = u(t^n)$ $\quad u^{n-1} = u(t^{n-1})$

$\underline{1^{st} \text{ order approximation:}}$ $\frac{y^n - y^{n-1}}{\Delta t} = f(t^{n-1}, y^{n-1}, u^{n-1})$

$$\boxed{y^n = y^{n-1} + \Delta t \, f(t^{n-1}, y^{n-1}, u^{n-1})}$$

Forward Euler. Explicit method because everything on right hand side is known.

Ex] $\frac{dy}{dt} = -y$ Exact solution: $y = e^{-t}$

$y(t=0) = 1$

Euler's method:

$y^n = y^{n-1} + \Delta t(-y^{n-1}) \Rightarrow y^n = (1-\Delta t)y^{n-1}$

$\Delta t = 0.001$

t	y
0	1
0.001	0.999
0.002	0.998
0.003	0.997

Ex] $\frac{d^2y}{dt^2} = -y$ $y(t=0) = 1$ $\dot{y}(t=0) = 0$; Exact solution: $y = \cos(t)$

State space: $x_1 = y$ $x_2 = \dot{y}$

$\frac{dx_1}{dt} = x_2$ $\frac{dx_2}{dt} = -x_1$

$x_1^n = x_1^{n-1} + \Delta t\, x_2^{n-1}$

$x_2^n = x_2^{n-1} + \Delta t(-x_1^{n-1})$

$\Delta t = 0.001$

t	x_1	x_2
0	1	0
0.001	1	-0.001
0.002	0.999999	-0.002
0.003	0.999997	-0.002999999